Journal of Statistical Physics, Vol. 4, No. 1, 1972

Questions and Answers

The Feedback Model of Systems Obeying Quantum Statistical Mechanics

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Received March 31, 1971

The aim of this paper is twofold. First, we shall derive the Fermi–Dirac (FD) and Bose–Einstein (BE) distributions from the classical Maxwell–Boltzmann (MB) distribution by introducing into the classical system the consequences of quantum mechanical indistinguishability in a direct and simple manner. Next, we go through a brief introduction to feedback systems and see how the FD and BE systems may be viewed as classical systems with appropriate feedback. We shall see that the resemblance to feedback systems is more than formal and that a feedback mechanism does exist in systems obeying quantum statistical mechanics.

KEY WORDS: Maxwell–Boltzmann distribution; Gibbs's canonical systems; Fermi, Dirac, and Bose–Einstein distributions; symmetrized and antisymmetrized states; positive and negative feedback systems.

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1. THE MAXWELL–BOLTZMANN DISTRIBUTION

To facilitate later discussion, let us first see how the MB distribution is worked out for classical distinguishable particles. Figure 1 shows a small system A in contact with a reservoir B. The combined system $A + B$ is isolated. Let us call the energies of the systems E_A and E_B. Let $\Omega_A(E_A)$ and $\Omega_B(E_B)$ be the number of microstates available to A and B, respectively, when their energies are E_A and E_B. Let A be in its ground state with energy $E_A{}^0$ and let B have the remaining energy, which we denote by $E_B{}^0$. ($E_B{}^0$ is *not* the ground-state energy of B, and in fact, it is the largest energy B can have.) If the ground state of A is degenerate, let it be in one of them. The number of microstates available to the combined system $A + B$ will be

$$\Omega_{A+B}(E_A{}^0) = 1 \times \Omega_B(E_B{}^0)$$

as a function of the energy of A. Let us now promote A to a state of energy ϵ_r over the ground state. The number of states available to the combined system is given by

$$\Omega_{A+B}(E_A{}^0 + \epsilon_r) = 1 \times \Omega_B(E_B{}^0 - \epsilon_r)$$

$$\log[\Omega_{A+B}(E_A{}^0 + \epsilon_r)] = \log[\Omega_B(E_B{}^0 - \epsilon_r)]$$

Assuming $\epsilon_r \ll E_B{}^0$, we may expand as follows:

$$\log[\Omega_B(E_B{}^0 - \epsilon_r)] = \log[\Omega_B(E_B{}^0)] - \beta\epsilon_r$$

$$\Omega_B(E_B{}^0 - \epsilon_r) = \Omega_B(E_B{}^0) \times e^{-\beta\epsilon_r}$$

where

$$\beta = \frac{\partial[\log \Omega_B(E_B)]}{\partial E_B}\bigg|_{E_B{}^0}$$

Postulating that all microstates of the combined, isolated system $A + B$

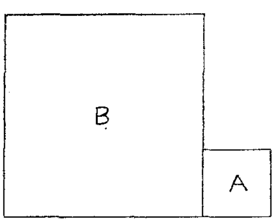

Fig. 1. System A in thermal contact with reservoir B.

are equally probable, we get the ratio of the probability P_A^r of A being in r to the probability P_A° of A being in the ground state as

$$\frac{P_A^r}{P_A^0} = \frac{\Omega_{A+B}(E_A^0 + \epsilon_r)}{\Omega_{A+B}(E_A^0)} = \frac{\Omega_B(E_B^0 - \epsilon_r)}{\Omega_B(E_B^0)} = Ce^{-\beta\epsilon_r}, \qquad C = \text{const.}$$

Now consider the problem of N distinguishable particles in a box. We want the probability P_r of a member being in a state r of energy ϵ_r above the ground state. We call this particle the system A and think of the others as constituting a reservoir B at temperature β. Due to the distinguishability, we may keep track of particle A and the distinction between A and B is maintained. We may rightly expect that A will have the same behavior as when it was isolated since the only difference now is that it exchanges heat with the members of B directly by collisions instead of via the conducting walls of the partition, which is an inconsequential distinction classically. So A will be in r with a probability $Ce^{-\beta\epsilon_r}$. Now, we may consider any member of the system to be A, and hence all members must behave alike. We then expect that the mean number of particles $n(r)$ in r will be given by

$$\begin{aligned} n(r) &= NP_r \propto e^{-\beta\epsilon_r} \\ &= e^{-(\alpha+\beta\epsilon_r)} \end{aligned}$$

where α satisfies

$$\sum_r e^{-(\alpha+\beta\epsilon_r)} = N$$

2. FERMI–DIRAC STATISTICS

We want to use an analysis similar to the one in the last section to find the mean number of fermions in state r in a box of N fermions. Let us first note a property of fermion systems:

$$\begin{aligned} n(r) &= \text{mean number of fermions in } r \\ &= (0 \times \text{probability of nonoccupancy of } r \text{ by any member}) \\ &\quad + (1 \times \text{probability of occupancy by any member}) \\ &= p_r \end{aligned}$$

Let us consider two boxes A and B separated by a conducting wall. Let us introduce a fermion called A in box A and the other $N - 1$ fermions in box B. By considering the effect of raising A over its ground state by ϵ_r on the number of microstates available to B, we get for the combined system $A + B$,

$$\Omega_{A+B}(E_A^0 + \epsilon_r) = \Omega_{A+B}(E_A^0)\, e^{-\beta\epsilon_r}$$

where $\beta = \partial[\log \Omega_B(E_B)]/\partial E_B$ once more. The fact that B is full of fermions may limit the accessible states of B to antisymmetrized ones, but β is still the energy derivative of the logarithm of accessible states. Further, the fact that A was a fermion was of no consequence. So we have the probability for a distinct fermion in box A, in thermal contact with a box full of fermions at temperature β, to be $Ce^{-\beta\epsilon_r}$. However, our problem is to find the energy distribution of fermions in a box. To find this, we shall use the method similar to the one used in the MB case, but with the indistinguishability of the fermions in mind. Our procedure is as follows:

1. We have the behavior of a fermion A in thermal contact with a box full of $N - 1$ fermions B.

2. We shall see what changes are to be expected when the partition between A and B is removed. We then modify 1 accordingly.

3. We play the trick of saying that if we have the behavior of one fermion, we have the behavior of them all, since any fermion may be treated as the one that was introduced from box A.

Let us begin with step 2. The first effect of removing the partition is that we lose track of A. *If the only effect of indistinguishability is that we cannot follow the motion of the particles individually, the following consequences are expected:*

(a) We cannot give probabilities for specific fermions to be in any given state. This is not an experimentally verifiable quantity.

(b) We can, however, speak of the probability *per* fermion to be in state r, $P_F{}^r$, in the following sense: If, on the average, there are $n(r)$ fermions in r in a box of N fermions,

$$P_F{}^r = n(r)/N$$

This probability is not attributed to any particular fermion, but to the entire collection, equally and uniformly. It may be experimentally verified by a measurement of $n(r)$.

(c) The fact that we cannot follow the motion of the fermion should not change its behavior when the partition is removed. If the isolated fermion (we use the term "isolated" to mean "when in box A" and not to mean thermally isolated) went to state r with a probability $ce^{-\beta\epsilon_r}$, it must be still doing so. Since all fermions are alike, we expect

$$n(r) \propto e^{-\beta\epsilon_r}$$

So, indistinguishability, *if* it merely means our inability to follow particles, leads to

$$n(r) = ce^{-\beta\epsilon_r}$$

However, indistinguishability in quantum mechanics has another important consequence for fermions: the exclusion principle. Whereas the isolated fermion in A could go to any state r of box A, it can go to a state r of box B only if r is free. If p_r is the probability that r is occupied by a member of B, the probability for the extra fermion going to r is now

$$P_F^{\ r} \propto e^{-\beta\epsilon_r}(1 - p_r)$$

But p_r is $n(r)$ for fermions. So

$$P_F^{\ r} \propto e^{-\beta\epsilon_r}[1 - n(r)]$$

Since all fermions have symmetric roles in that any one of them may be considered the one that was let in, they all follow the same distribution $P_F^{\ r}$. So

$$n(r) = NP_F^{\ r} \propto e^{-\beta\epsilon_r}[1 - n(r)]$$

$$n(r) = \frac{c'e^{-\beta\epsilon_r}}{1 + c'e^{-\beta\epsilon_r}} = \frac{1}{e^{\alpha+\beta\epsilon_r} + 1}, \qquad c' = \text{const}$$

To those readers who feel our arguments lack rigor, we offer the following lengthy, but hopefully more rigorous, proof. We shall count states, which is the surest method. Consider a system of N fermions. Say we want to describe a state in which there is one fermion in state i, one in j, etc. We first pretend that they are distinguishable and write a direct tensor product

$$| \psi\rangle = | 1\rangle_i | 2\rangle_j \cdots | N\rangle_s$$

where the numerals label the particles and the subscripts label the states. To get an antisymmetric state for fermions, we operate on this with the antisymmetrization operator:

$$A_N = (N!)^{-1/2} \sum_p (-1)^p \, \mathbb{P}$$

where the sum includes all the $N!$ permutations $\mathbb{P}$, and p is the number of transpositions in $\mathbb{P}$. So

$$A_N | \psi\rangle = | \psi\rangle_a$$

The subscript a is to remind us that the state vector is antisymmetric. A_N also has the following interesting property. If we formed an antisymmetric state $| \psi\rangle_x$ with x fermions, and another $| \psi\rangle_y$ with y fermions, the state for the combined system is

$$A_{x+y} \, | \psi\rangle_x \, | \psi\rangle_y = | \psi\rangle_a$$

If a one-particle state r had been occupied both in $| \psi\rangle_x$ and $| \psi\rangle_y$, then

$$A_{x+y} \, | \psi\rangle_x \, | \psi\rangle_y \equiv 0$$

which is the Pauli principle at work.

Let us now return to our problem. Consider a system B made of $N - 1$ fermions forming a reservoir at temperature β. This implies $\beta = \partial[\log \Omega_B(E_B)]/\partial E_B$, where $\Omega_B(E_B)$ is the number of states that can be formed by the $N - 1$ fermions at the given energy, with the exclusion principle applied among the $N - 1$ fermions. Consider now the introduction of any distinguishable member A, say a boson. The energy of $A + B$ is constant. If A goes to a state r, with energy ϵ_r above the ground state, given by $| \psi_A{}^r\rangle$, the states available to the $N - 1$ fermions goes as $\Omega_r \propto e^{-\beta\epsilon_r}$. If one such $(N - 1)$-fermion state is called $| \psi_{N-1}^r\rangle$, the state of the combined system is $| \psi\rangle = | \psi_A{}^r\rangle \, | \psi_{N-1}^r\rangle$, and there are Ω_r such states in this situation, with A in r. However, if A were a fermion, we must properly antisymmetrize and get for the combined system

$$| \psi\rangle_a = A_N \, | \psi_A{}^r\rangle \, | \psi_{N-1}^r\rangle$$

However, all states formed using a $| \psi_{N-1}^r\rangle$ with a particle from B in r will be nullified by A_N when combined with $| \psi_A{}^r\rangle$. If, out of the Ω_r states, $\Omega_r{}^r$ had a particle of B in r, the number of allowed states goes as

$$\propto e^{-\beta\epsilon_r} \times [1 - (\Omega_r{}^r/\Omega_r)]$$

But, if, out of Ω_r states, $\Omega_r{}^r$ had a particle in r, $\Omega_r{}^r/\Omega_r$ is the probability that r was occupied in B. So the number of allowed states goes as

$$\propto e^{-\beta\epsilon_r}(1 - p_r) \propto e^{-\beta\epsilon_r}[1 - n(r)]$$

Assuming all properly antisymmetrized states are equally probable, the probability for the fermion going into r goes as

$$P_F{}^r \propto e^{-\beta\epsilon_r}[1 - n(r)]$$

Since any fermion may be called the newly introduced fermion, the above calculation is true for all. Therefore

$$n(r) = NP_F{}^r \propto e^{-\beta\epsilon_r}[1 - n(r)]$$

$$n(r) = ce^{-\beta\epsilon_r}/1 + ce^{-\beta\epsilon_r} = 1/(e^{\alpha+\beta\epsilon_r} + 1), \qquad c = e^{-\alpha} = \text{const}$$

The reader must note how $n(r)$ depends on itself in a self-destructive manner. This is the feedback effect that we will discuss later.

Our analysis would be incomplete if we did not point out and justify an approximation we have made. We said

$$\Omega_r^r/\Omega_r = p_r = n(r)$$

where Ω_r is the number of states available to B when A has an energy ϵ_r over the ground state E_A^0, and B has energy ϵ_r below its maximum; Ω_r^r is the number of these states with r occupied; p_r is the probability that r is occupied; and $n(r)$ is the mean number of particles of B in r. This approximation is not strictly correct.

The values for $n(r)$ and p_r that we have calculated are valid at a particular energy of B, $E_B^0 - \epsilon_r$. At another energy ϵ_s of A, B will have an energy $E_B^0 - \epsilon_s$ and number of states Ω_s, and of these, Ω_s^r will have r occupied, and the ratio of Ω_s^r to Ω_s will be different. The real mean of the number of particles in r must be calculated by taking the weighted average of these means. However, if it is true that

$$\Omega_r^r/\Omega_r = \Omega_s^r/\Omega_s = \cdots$$

$n(r)$ may be calculated at any energy. In other words, we must show that the mean occupancy of a state in a reservoir does not change when its energy is changed in the order of a single particle energy. We shall do so shortly, but first point out that it is not too obvious. In general, the effect of energy changes in the order of ϵ_r are not ignorable. After all, the canonical distribution $ce^{-\beta\epsilon_r}$ was derived by considering the change in the number of reservoir states when its energy was changed by ϵ_r. Further, since the number of states is decided by the number of possible distinguishable arrangements of the particles among the states, this cannot change without a change in the $n(r)$. This is true, but we shall see that the change in the $n(r)$ is utterly negligible.

Our proof will be on hindsight. Our calculation assuming constancy of $n(r)$ gave

$$n(r) = 1/(e^{\alpha+\beta\epsilon_r} + 1)$$

and this is strictly a function of β and will change if β does. But since B is a reservoir, its β will not change by any sizable amount as its energy is changed by ϵ_r. We shall use this same idea when we come to the problem of bosons.

3. THE BOSE–EINSTEIN DISTRIBUTION

We shall derive the BE distribution in a manner that may at first sight seem different from the way we derived the FD distribution. We shall later on show the equivalence of the two methods.

As a prelude, let us state an important and rather well-known property of bosons, which we have proved in the appendix for completeness. The property follows from the requirement that bosons be described by a symmetric wave function, and is the following: If P_{if} is the probability that an isolated boson in state i goes to state f, this is enhanced by a factor $(n_f + 1)$ if there are already n- bosons in f. We shall refer to this as the antiexclusion principle.

An analogous principle exists for fermions and runs as follows: If P_{if} is the probability that an isolated fermion goes from state i to state f, this is modified by a factor $(1 - n_f)$ if there are already n_n fermions in f. This is of course Pauli's principle, restated to resemble the antiexclusion principle. These are properties peculiar to quantum systems and follow from the fact that identical particles in quantum mechanics are described en masse by a single, symmetrized or antisymmetrized wave function, which tends to correlate their behavior. This intrinsic property must be incorporated in any many-body problem. We shall do so explicitly for bosons and point out how we have already done so in our fermion calculations.

At the risk of sounding repetitious, but with the hope of adding to the clarity we shall begin at a basic level, and proceed to find the mean number $n(r)$ of bosons in a state r, in a box of N bosons. Defining once more the probability per boson to be in r as

$$P_B{}^r = n(r)/N$$

our problem reduces to finding this.

Consider an isolated boson A in box A, in contact with a box B containing $N - 1$ bosons. As A goes to a state r of energy ϵ_r over the ground state, the number of accessible states of B falls as $Ce^{-\beta\epsilon_r}$, where β is the temperature of B. The state of A is described by a state vector $|\,\psi_A{}^r\rangle$, while B is described by $|\,\psi^r_{N-1}\rangle_i$, which is a symmetrized $(N-1)$-particle state vector indicating that B is in some state i while A is in r. The combined system is given by

$$|\,\psi\rangle = |\,\psi_A{}^r\rangle\,|\,\psi^r_{N-1}\rangle_i$$

The number of these states as a function of ϵ_r goes as $\alpha e^{-\beta\epsilon_r}$.

The probability that A is in r, $P_A{}^r$, is the sum of the probabilities for the different ways in which this can happen,

$$P_A{}^r = \sum_i P^r_{A,B_i}$$

where P^r_{A,B_i} is the probability that A goes to r when B is in i. The sum is over all states i that B can be in, when A has energy ϵ_r over the ground state. Since even quantum mechanics does not correlate the motion of A with B when A is isolated and distinguishable, and since quantum mechanics does not intrinsically favor any one particle state r of A, the terms in the sum are all equal. So we get

$$P_A{}^r = \text{number of terms in the sum} \propto e^{-\beta\epsilon_r}$$

We now want to let A into box B and ask for the probability that the extra boson is in a state r of box B. At this point, we may raise a point that may well have been brought up when doing the fermion calculation also. Can we speak of the probability of the extra boson being in r, when it is not distinguishable from the rest? Yes, in the following sense. If A were distinguishable, a typical state would be $|\,\psi\rangle = |\,\psi_A{}^r\rangle\,|\,\psi^r_{N-1}\rangle_i$, which tells us that the extra particle A is in r. If A were indistinguishable, we would symmetrize this with the symmetrizing operator S to get

$$S\,|\,\psi\rangle = |\,\psi\rangle_S$$

Now, $|\,\psi^r_{N-1}\rangle_i$ is an $(N-1)$-particle state that tells us where the $N-1$ particles are, without naming them, of course. $|\,\psi\rangle_S$ is an N-particle state that tells what the N particles are doing. A comparison of the two will tell us what the "extra" particle is doing: it is in r. We call such a state as one with the extra particle in r.

We then have the probability for this to be

$$P_A{}^r = \sum_i P^r_{A,B_i}$$

In summing, we no longer assume P^r_{A,B_i} is independent of i. The antiexclusion principle tells us that if there are two states i and j of B, i with no particles in r and j with $n_j{}^r$ particles in r,

$$P^r_{A,B_j} = (n_j{}^r + 1)\,P^r_{A,B_i}$$

We therefore give each term a weight $(n_j{}^r + 1)$ so that

$$P_A{}^r \propto \sum_j\,(n_j{}^r + 1)$$

The sum of $n_j{}^r + 1$ over the states of B is clearly equal to the mean of $(n_j{}^r + 1)$ times the number of terms in the sum. Since the number of terms goes as $\propto e^{-\beta\epsilon_r}$,

$$P_A{}^r \propto e^{-\beta\epsilon_r}[1 + n(r)]$$

where we have called the mean number of n_j^r as $n(r)$. Since any boson may be called the newcomer, they all obey the same statistics, and so

$$n(r) = NP_A^{\ r} = ce^{-\beta\epsilon_r}[1 + n(r)]; \qquad n(r) = 1/(e^{\alpha+\beta\epsilon_r} - 1)$$

A number of points need to be discussed. We first note that $n(r)$ is the mean number calculated at one energy of B, i.e., $E_B{}^0 - \epsilon_r$. But as in the fermion case, this is alright, since the occupancy numbers of the reservoir do not change due to energy changes like ϵ_r. Second, the reader may feel that in assigning different weights like $(n_j^r + 1)$ to different microstates of the extra particle contradicts the statistical postulate of equal probabilities. This is not so. We are weighting here the probabilities for a single member to go to a state when the others are assumed to be in some configuration, while the postulate refers to the entire system. Is it possible for the members of a system to show preference for certain one-particle microstates while the entire system is equally likely to be in any of its system microstates? Yes. The classical canonical distribution is an example in which individual members go for a state r with probability $\alpha e^{-\beta\epsilon_r}$, while the system as a whole goes to all its states with equal probability.

Lastly, one may ask, "How come we did no such weighting for the fermions? Did we not give equal status to all allowable states?" The answer is that we did perform a rather drastic weighting in calling certain states "allowed" and the others "disallowed." Of the total of Ω_r states, we threw away $\Omega_r{}^r$ and gave the others equal weight. The way to interpret this is to say that we gave the states j in which r was occupied a weight of $1 - n_j^r = 0$ since $n_j^r = 1$, and those with r free a weight of $1 - n_j^r = 1$ since $n_j^r = 0$. For fermions, the weighting process is a binary "yes" or "no" process.

To the reader who feels that we could have saved time and space if we had done the fermion calculations with weights the first time itself, we offer the following explanation. This was done on purpose. We wanted to emphasize in this paper three main ideas, and we wanted to do this step by step. In the first section, we wanted to point out the generality of the canonical distribution, in that it describes the probabilities for an isolated fermion, boson or boltzon, to be in a state r when in thermal contact with a reservoir made of like or unlike particles. In the next section, we wanted to show how this knowledge, when combined with the knowledge of the effect of removing the partition, led to the behavior of particles when sharing the same space as other members of the same species, i.e., to the occupancy numbers. It was easy to do this for fermions, since the only effect of removing the partition was to exclude certain states. Then we wanted to bring in the concept of weights, which was more general, through the *BE* calculation.

In the next section, we shall see that the quantum systems may be derived from classical systems by assuming suitable feedbacks. Once more we shall see that the idea of feedback had been unwittingly used by us in our derivations. To the reader who asks why we did not use feedback ideas right away, we offer the same explanation as above.

4. THE FEEDBACK MODEL

Let us first get acquainted with some basic concepts in feedback systems. Consider an amplifier of gain A, as shown in Fig. 2(a). The output O is related to the input I by

$$O = AI$$

Imagine that a fraction B (called the feedback factor) of the output is fed back to the input, so as to oppose it, via the mixer M. Such a system is called a negative feedback (NFB) system, and is shown in Fig. 2(b). The negative sign at the mixer indicates NFB. What is the effective gain of the amplifier? The output O depends on the input, which depends on the output due to the feedback. We can, however, break this chain by noting that always

$$\begin{aligned} O &= A \times (\text{what goes into the amplifier}) \\ &= A \times (I - BO) \\ O &= [A/(1 + AB)]I = A'I \end{aligned}$$

A' is called the effective gain. Negative feedback systems are very stable in the effective A' as A varies. This is because the feedback opposes the input less if the output tries to fall and opposes the input more when the output tries to rise.

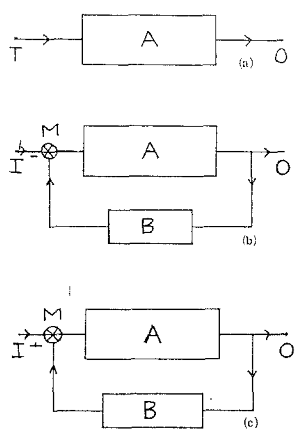

Fig. 2. (a) Open-loop system. (b) Negative feedback system. (c) Positive feedback system.

We may also consider a system where the feedback aids the input. This is called a positive feedback (PFB) system and is shown in Fig. 2(c). The effective gain here is

$$A' = A/(1 - AB)$$

A' is very unstable since the feedback aids the output via the input when it tries to go up, and accelerates its decline.

We urge the reader, if has not already done so, to note the similarity between the factors

$$A, \qquad A/(1 + AB), \qquad A/(1 - AB)$$

describing open-loop, NFB, and PFB systems and the factors

$$ce^{-\beta\epsilon_r}; \qquad ce^{-\beta\epsilon_r}/(1 + ce^{-\beta\epsilon_r}), \qquad ce^{-\beta\epsilon_r}/(1 - ce^{-\beta\epsilon_r})$$

describing MB, FD, and BE systems, respectively. This hints at the possibility of considering FD systems as MB systems with NFB and BE systems as MB systems with PFB in some sense. We shall see how this can be done.

The classical system is described by the formula

$$n(r) = ce^{-\beta\epsilon_r}$$

where $n(r)$ is the mean number of particles in a one-particle quantum state r. If we now let r stand for an energy (not *a* state but *all* states of energy ϵ_r), and if the degeneracy of r is denoted by g_r, the number N_r in the level r is

$$N_r = g_r ce^{-\beta\epsilon_r} = g_r n(r)$$

We represent such a system schematically by Fig. 3(a). It is supposed to repre-

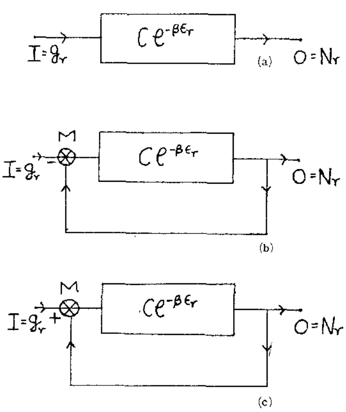

Fig. 3. (a) Maxwell–Boltzmann system (schematic). (b) Fermi–Dirac system (schematic). (c) Bose–Einstein system (schematic).

sent the following: The probability for a particle to be in r is decided by the energy and degeneracy of r. This is independent of what the other particles are doing, particularly on how many are already in r. This is implied in the absence of the feedback loop. The factor $ce^{-\beta\epsilon_r}$ comes from a counting of possible states for the reservoir. This depends only on the temperature of the reservoir and not on its composition.

Say we apply to this system a NFB with the feedback factor equal to uniy. Such system is represented in Fig. 3(b). It follows that

$$N_r = (g_r - N_r)\, ce^{-\beta\epsilon_r} = g_r[1 - (N_r/g_r)]\, ce^{-\beta\epsilon_r} = g_r[1 - n(r)]\, ce^{-\beta\epsilon_r}$$

$$N_r/g_r = n(r) = g_r ce^{-\beta\epsilon_r}[1 - n(r)]$$

$$n(r) = ce^{-\beta\epsilon_r}/(1 + ce^{-\beta\epsilon_r}) = 1/(e^{\alpha+\beta\epsilon_r} + 1)$$

We have shown here explicitly the steps so that the similarities with our earlier derivation are transparent. The reader should see for himself the physical interpretation of the steps. We note that, being negative feedback systems, the fermion systems have very stable occupancy numbers as the temperature varies. This must be familiar to those who deal with fermion gases and see how the populations of the levels change with temperature with great reluctance.

Consider next the problem of applying unity PFB to a MB system, as in Fig. 3(c). The following are the implications: (1) The probability for a member to be in r is enhanced by the presence of other members in r. (2) The mean occupancy number of energy r depends on itself in a self-aiding manner.

These are characteristic of boson systems. Let us see how it works out. We have, from Fig. 3(c),

$$N_r = (g_r + N_r)\, ce^{-\beta\epsilon_r} = g_r ce^{-\beta\epsilon_r}[1 + (N_r/g_r)] = g_r ce^{-\beta\epsilon_r}[1 + n(r)]$$

$$N_r/g_r = n(r) = ce^{-\beta\epsilon_r}[1 + n(r)]$$

$$n(r) = ce^{-\beta\epsilon_r}/(-ce^{-\beta\epsilon_r} + 1) = 1/(e^{\alpha+\beta\epsilon_r} - 1)$$

Boson populations can get very unstable as β changes (condensation).

5. CONCLUSIONS

We sum up our arguments as follows. Classical systems consist of distinguishable particles, which may be described individually. Their motions are not therefore correlated. The probability for a particle going to a given state is not affected by the occupancy of that state. Such a system obeys MB statistics and is represented by an open-loop system.

Quantum systems consist of indistinguishable particles which have to be described en masse by properly symmetrized or antisymmetized state vectors. Their motions are therefore correlated even though they are assumed to be noninteracting (in the sense that the interaction Hamiltonian is utterly negligible). There are two kinds of correlations. Fermions are negatively correlated, in the sense that the presence of a particle in a state r prohibits the entry of another particle. The bosons are positively correlated in the sense that the presence of bosons in a state makes the arrival of other bosons more likely. This is a feature intrinsic in the quantum mechanics. Fermions are described as NFB systems as compared to uncorrelated MB systems, while bosons are described as PFB systems as compared to MB systems. In both cases, the feedback factor is unity.

Of course, both FD and BE systems may be described as open-loop systems, as in Fig. 4, without any mention of feedback or correlation. But we like the think of them as uncorrelated systems with appropriate feedback for the following reasons:

1. The canonical distribution is a very general one ($ce^{-\beta\epsilon_r}$). It describes the behavior of isolated fermions, bosons, or boltzons when they interact with a reservoir through a partition. However, when the isolation is ended by the removal of the partition, the motion of the particle in general gets correlated with those in the reservoir. The correlation exists only if the reservoir has the same kind of particles as the isolated one. The different kinds of correlation and the strengths of correlation (magnitude of the feedback factor) are transparently contained in the feedback loop.

2. In fact, there *is* a correlation or feedback. Particles seem to be aware of what the others are doing. Fermions seem to obey traffic laws. Bosons seek the company of other bosons. While all combinatorial calculations for the distributions take explicit note of the tendency of fermions to avoid each other by counting states in harmony with the exclusion principle, they do not explicitly incorporate the tendency of bosons to stick together. We feel our approach treats fermions and bosons in a symmetric manner.

3. We feel that the macroscopic equivalents of the exclusion and antiexclusion principles are provided by the statements that fermions are

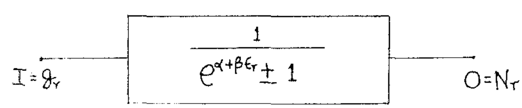

Fig. 4. Open-Loop model of quantum systems.

NFB systems with unity feedback, and bosons are PFB systems with unity feedback.

4. We sincerely feel that the feedback approach will lead beyond the results derived in this paper if an attempt is made by those better versed than the present author in feedback systems, statistical mechanics, or both. The paper is written with that hope.

APPENDIX

We plan to show here that if P_{if} is the probability that an isolated boson goes to state $|f\rangle$ from state $|i\rangle$, this is enhanced to $(n_f + 1)\, P_{if}$ if there are already n_f bosons present in $|f\rangle$. Consider three particles in states $|1\rangle$, $|2\rangle$, and $|3\rangle$ as shown in Fig. 5. We want to investigate the probability of their going to state $|f\rangle$.

Case 1. Assume they are distinguishable. We christen them A, B, and C. The initial state is given by the tensor product

$$|\psi_i\rangle = |1\rangle_A\, |2\rangle_B\, |3\rangle_C$$

where the numerical labels are for the states and the letters for the particles. The final state is

$$|\psi_f\rangle = |f\rangle_A\, |f\rangle_B\, |f\rangle_C$$

and the amplitude for the process $i \to f$ is

$$\begin{aligned}\langle\psi_f \mid \psi_i\rangle &= {}_A\langle f \mid {}_B\langle f \mid {}_C\langle f \mid \,|\, |1\rangle_A\, |2\rangle_B\, |2\rangle_C \\ &= \langle f \mid 1\rangle_A \langle f \mid 2\rangle_B \langle f \mid 3\rangle_C = \alpha\beta\gamma\end{aligned}$$

where $\alpha = \langle f \mid 1\rangle_A$, etc. Clearly, $\langle f \mid 1\rangle_A = \langle f \mid 1\rangle_B$, etc., since the amplitude to go from $|i\rangle$ to $|f\rangle$ is a function of $|i\rangle$ and $|f\rangle$ and not the particle.

The probability for the above case of distinguishable particles is

$$P_{if} = \alpha^2\beta^2\gamma^2$$

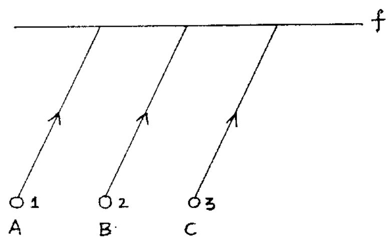

Fig. 5. Arrival of three bosons to a state f.

Case 2. Let the particles be indistinguishable bosons. We merely know that in the initial state, there are three bosons, one each in $|1\rangle$, $|2\rangle$, and $|3\rangle$, and the legitimate symmetrized state is given by

$$|\psi_i\rangle = (3!)^{-1/2}(|1\rangle_A|2\rangle_B|3\rangle_C + |2\rangle_A|1\rangle_B|3\rangle_C + \cdots)$$

The final state $|\psi_f\rangle = |f\rangle_A|f\rangle_B|f\rangle_C$ is already symmetrized. The amplitude to go from i to f is

$$\begin{aligned}\langle\psi_f|\psi_i\rangle &= (3!)^{-1/2}(\langle f|1\rangle_A\langle f|2\rangle_B\langle f|3\rangle_C + \langle f|2\rangle_A\langle f|1\rangle_B\langle f|3\rangle_C + \cdots)\\ &= (3!)^{-1/2}\,3!\,\alpha\beta\gamma = (3!)^{1/2}\,\alpha\beta\gamma\end{aligned}$$

Labeling the probability for the bosons and the distinguishable cases appropriately, we have

$$P_{if}(B) = 3!\,\alpha^2\beta^2\gamma^2 = 3!\,P_{if}(D)$$

This means that if there are three bosons in states $|1\rangle$, $|2\rangle$, and $|3\rangle$, the probability for them to go to $|f\rangle$ is $3!$ times as much as it would have been if they had been distinguishable. For N particles, it follows that

$$P^N(B) = N!\,P^N(D)$$

and similarly for $N+1$ particles,

$$P^{N+1}(B) = (N+1)!\,P^{N+1}(D)$$

where we have dropped the subscript *if*. But

$$\begin{aligned}P^{N+1}(B) &= P^N(B) \times [\text{probability for the arrival of the } (N+1)\text{th boson given } N \text{ are already there}]\\ &= P^N(B) \times P^{N+1}(B)_{\text{arr}}\end{aligned}$$

Similarly for the distinguishable case,

$$P^{N+1}(D) = P^N(D) \times P^{N+1}(D)_{\text{arr}}$$

so that

$$\begin{array}{ccc} P^{N+1}(B) = P^N(B) \times P^{N+1}(B)_{\text{arr}} & = & (N+1)!\,P^{N+1}(D)\\ \downarrow & & \downarrow\\ P^N(D) \times N! \times P^{N+1}(B)_{\text{arr}} & = & (N+1)! \times P^N(D) \times P^{N+1}(D)_{\text{arr}}\end{array}$$

Therefore,

$$P^{N+1}(B)_{\text{arr}} = (N+1)\,P^{N+1}(D)_{\text{arr}}$$

The arrival of the $(N+1)$th particle, if we are dealing with bosons, is $(N+1)$ times as probable as the arrival of the $(N+1)$th particle if we were dealing with distinguishable particles. Noting that the first particle came as an isolated and distinguishable particle, we have the result

$$P^{N+1}(B)_{\text{arr}} = (N+1)\, P^{1}(B)_{\text{arr}}$$

This may be restated in a manner that will be more appropriate to our discussions : If there are two "closely lying" states r and s in the sense that an isolated boson would go to either with equal probabilities, the probability for it to go to r will be $(n_r + 1)$ times the probability for it to go to s if r has already n_r particles in it and s is free. These results may be proved very easily using the raising and lowering operators of second quantized theory.

ACKNOWLEDGMENT

I have great pleasure in thanking my friend Rajaram Nityananda for interesting and helpful discussions.